AP CALCULUS (BC) NAME \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_

Chapter 9 – Infinite Series Date \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_

**Section 9.1 – Power Series**

1) What is the difference between an *infinite sum* and a finite sum?

2) Infinite Series

Definition:

An infinite series is an expression of the form:

**a1 + a2 + a3 + … + an + …**, or 

where **an** is the nth term of the series.

1. Partial Sums of the Series – a sequence of real numbers defined by a finite sums of the series.

**s1 = a1**

**s2 = a1 + a2**

**s3 = a1 + a2 + a3**

**sn = a1 + a2 + a3 + … + an =** 

If **lim sn = S**, then the *infinite series* converges to the sum **S**, and we

**n 🡪 ∞**

write: **a1 + a2 + a3 + … + an + … =** = **S**. Otherwise the series diverges.

1. Examples 1 and 2 (page 474)
2. Theorem

If an *infinite series* converges, then **lim ak = 0.**

**k 🡪 ∞**

And if **lim ak ≠ 0**, then the *infinite series* must diverge!

**k 🡪 ∞**

{this is more important than it looks!}

2) Geometric Series

**a + ar + ar2 + ar3 + … + arn-1 + … =** 

converges to the sum **a/(1 – r)** if **│r│ < 1**; it diverges if **│r│ > 1**.

If the geometric series converges, then **-1 < r < 1** is the **interval of convergence**.

a) Example 3 (page 475)

b) Constructing a Special Geometric Series

If we let **a = 1** and **x = r**, such that **│x│ < 1**, then the GS looks like this…

**1 + x + x2 + x3 + … + xn + … = 1/(1 – x).**

The importance here is that this series represents a function!

c) Investigation – using the GC, graph the various partial sums of this series, and see

how well each one approximates the function **y = 1/(1 – x)** on (-1, 1).

3) Power Series

An expression of the form: = **c0 + c1x + c2x2 + … + cnxn + …**

is a *power series centered at x = 0*. An expression of the form:

= **c0 + c1(x – a) + c2(x – a)2 + … + cn(x – a)n + …**

is a *power series centered at x = a.*

The GS **1 + x + x2 + … + xn + … =**  is a power series centered at x = 0, and converges on (-1, 1) {which is also *centered* at x = 0}

Something which will be confirmed later is that *power series* will either:

* converge for all x;
* converge on a finite interval whose center is the center of the interval; or
* converge only at the center!

4) Power Series and Other Functions – Exploration 1 (page 477)

Recall: The function  is represented by= **1 + x + x2 + x3 + … + xn + …**

on the interval (-1, 1).

**Homework 9.1a: page 481 # 1ab, 2abc, 3, 4, 7, 9, 11, 14, 15, 18**

5) Power Series and Calculus

1. Discovering a Power Series by Differentiation

Find a PS for: . Begin with the PS for: .

Graph some of the partial sums versus the function.

What about the **interval of convergence**?

See Theorem 1 (page 478) – Term by Term Differentiation

1. Discovering a Power Series by Integration

Find a PS for ln(1 + x). Begin with PS for \_\_\_\_\_\_\_\_\_\_\_\_.

Graph some of the partial sums versus the function.

See Theorem 2 (page 479) – Term by Term Integration.

1. Investigate what happens to both PS at x = 1.

6) Exploration 2 (page 480) – Determine a PS for tan-1x.

7) Exploration 3 (page 480) – Working backwards?

1. Determining Functions Which a Given Power Series Represents

Let: 

**Homework 9.1b: page 481 # 23, 24, 27, 30, 31, 39a, 48, 54, 55, 61, 63, 72**

**Section 9.2 – Taylor Series**

1) Exploration 1 (page 484) – Group Work

Construct a polynomial (4th degree), , which meets the following conditions:

**P(0) = 1; P’(0) = 2; P”(0) = 3; P’’’(0) = 4; P’’’’(0) = 5**

Based on the results of this exploration, we now formalize the process for constructing polynomials that approximate “other” functions by emulating the functions behavior at x = 0.

2) Use the above technique to construct a 4th degree polynomial (aka a Taylor polynomial of

degree 4) to approximate, **f(x) = ln (1 + x)** at **x = 0**.

**Homework 9.2a: page 492 # 2, 23, 39**

3a) Construct a Power Series for **sin x** at **x = 0**. Graph **Pn(x)** vs. **sin x** and BEHOLD!!!

**Homework 9.2b: page 492 # 3, 15**

3b) Construct a Power Series for **cos x** at **x = 0**.

Discussion:

* These T-polynomials are not new – why not?
* **Sin x** and **cos x** are what type of functions? How does that relate to their T-polynomials?

4) MacLauren Series (Taylor Series centered at x = 0)

MacLauren Series (Taylor Series centered at x = 0):



**Homework 9.2c: page 492 # 5, 6, 24, 31, 32, 40, 43, 44**

5) Taylor Series (centered at x = a)

Taylor Series centered at x = a:



**Homework 9.2d: page 492 # 13, 18, 21, 26, 42**

**Section 9.3 – Taylor’s Theorem**

Although Taylor series exist whose infinite sums are *exactly* equal to certain functions, such as sin x, in reality it is the Taylor polynomials that “do all the work.” As previously noted, even the most powerful calculators and computers use Taylor polynomials to determine approximate values for these functions.

The only questions that needs to be made in determining which order Taylor polynomial is required…is how accurate must the approximations be?

1) Problem Statement: What order Taylor polynomial is needed to approximate sin x within

.0001 anywhere on [-π, π]?

This problem, and its solution, opens up the discussion of an important topic in convergent series and their value as evaluation tools. How can we calculate these errors which arise from using an approximating Taylor polynomial? And how can we keep them within specified bounds? The errors in using approximating polynomials are called, *truncation errors,* since they result from truncating the infinite series down to some finite polynomial of degree n.

2) Geometric Series – Determine the truncation error that arises from using 

to approximate  on (-1, 1). (If stuck, see page 496, Example 2)

**Homework 9.3a: page 500 # 14**

Although this example suggests how we might determine truncation error for any geometric series; and demonstrates that we can find an exact value for that error, it should be clear that not all series are geometric. With this in mind we proceed to try to get a handle on truncation errors in non-geometric series.

3) **Taylor’s Theorem with Remainder** (Theorem 3 – page 496) – Stated without proof

If *f* has derivatives of all orders in an open interval *I* containing *a*,

then for each positive integer *n* and for each *x* ε *I*,

where

 for some c between a and x.

This theorem tells us how to construct the approximating polynomial for a function with derivatives of all orders; and it also gives a formula for the error generated using that polynomial. *f(x)* is Taylor’s formula; and *Rx(x)* is the remainder of order n or the error term for the approximation *Pn(x)* over *I*. *Rn(x)* is also called the Lagrange form of the remainder.

**Homework 9.3b: page 500 # 1**

Taylor’s Theorem gives us a method for proving convergence (as outlined in the box below):

We say that a Taylor series generated by *f* at *x = a* converges to *f* on *I*, if *Rx(x) 🡪 0* as *n 🡪 ∞*; and we write: .

4) Using Taylor’s Theorem with Remainder to Prove Convergence

1. Prove:  converges to *sin x* for all *x ε R*. Show that *Rx(x) 🡪 0* as *n 🡪 ∞*.
2. Prove:  converges to *cos x* for all *x ε R*.

***The method of proof used above is so powerful that we generalize its use and state it as the following theorem:***

5) Remainder Estimation Theorem (Theorem 4 – page 498)

If there are positive constants *M* and *r* such that

 for all *t* between *a* and *x*,

then the remainder Rn(x) in Taylor’s Theorem satisfies the inequality

.

If these conditions hold for every *n* and all the other conditions of Taylor’s Theorem are

satisfied by *f*, then the series converges to *f(x).*

**Homework 9.3c: page 500 # 7, 15, 33, 35, 44**

6) Exploration 2 (page 499) – Euler’s Formula

**Homework 9.3d: page 502 # 1 – 4; {HINT on #4 – what type of series is f(x) in #4?}**

**Section 9.4 – Radius of Convergence**

Since a series is an infinite sum and a convergent series is a series which converges to a particular sum, then in reality a convergent series is a number; while a divergent series is not!

In this section we turn our attention to determining for what values of x a particular power series converges. Determining the *Interval of Convergence* amounts to two activities: 1) finding the radius of convergence; and 2) then determining what happens at the endpoints. We will first turn our attention to finding the radius of convergence.

1a) Recall that if a series is geometric (**see example 1 – page 503**), then we can easily determine

the values of x for which the GS will converge.

1b) According to Theorem 5 (page 504) there are only 3 possibilities for any power series of the

form: ; either:

* The series only converges at the center ***x = a***; or
* The series converges for all ***x ε R***; or
* The series converges on some finite interval centered at ***x = a***.

NOTE: ***R*** is the radius of convergence and ***a – R < x < A + R*** is the interval of

Convergence.

2) Tests (Methods) for Determining Convergence/Divergence of a Power Series

a) nth-Term Test (Theorem 6 – page 504): If , then  diverges.

Example: ; Divergent?

**Homework 9.4a: page 511 # 38**

b) Direct Comparison Test (Theorem 7 – page 505):

Let be a series with non-negative terms:

i) converges if there exists a convergent series with **an ≤ cn** for all

**n > N;**

ii) diverges if there exists a divergent series with **an ≤ dn** for all

**n > N.**

See Example 2 – page 505

**Homework 9.4b: page 511 # 4, 33**

Definition – Absolute Convergence:

If the series  of absolute values converges, then  **converges absolutely**.

Theorem 8 – Absolute Convergence 🡪 Convergence: If  converges, then converges.

See Example 3 – page 506

**Homework 9.4c: page 511 # 5**

c) Ratio Test (Theorem 9 – page 507)

Let be a series with positive terms, and with  then,

1. the series converges if ***L* < 1**;
2. the series diverges if ***L* > 1**;
3. the test is inconclusive if ***L* = 1**.

NOTE: The ratio actually gives us a commonly-used method for determining the radius of convergence for a particular power series; in addition, when applying the ratio test we actually use absolute convergence which is a stronger result than necessary.

See Exploration 1 (page 508)

See Examples 4, 5, and 6 (pages 508 – 509)

**Homework 9.4d: page 511 # 9, 12, 14, 17, 55 – 59, 64**

3) Telescoping Series – see Example 7 (page 510)

**Homework 9.4e: page 511 # 49, 50, 60**

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**Section 9.5 – Testing Convergence at Endpoints**

1) The Integral Test

Let ***{an}*** be a sequence of positive terms. Suppose that ***an = f(n),*** where ***f*** is a continuous positive, decreasing function of ***x*** for all ***x ≥ N*** (*N* a positive integer). Then the series and the integral  either both converge or diverge.

See Example 1 (page 513)

2) The Harmonic Series and other p-series

See Exploration 1 (page 514)

**Homework 9.5a: page 523 # 1**

3) The Limit Comparison Test (LCT)

Suppose that ***an > 0*** and ***bn > 0*** for all ***n > N*** (N, a positive integer).

1. If  where ***0 < c < ∞,*** then  and  both converge or both diverge.
2. If  and  converges, then  converges.
3. If  and  diverges, then  diverges.

See Example 3 (page 515)

**Homework 9.5b: page 523 # 5, 6**

4) Alternating Series

a. Alternating Series Test – Leibniz’s Theorem

The series  converges if the following 3

conditions are all satisfied:

1. each ***un*** is positive;
2. ***un ≥ un+1*** for all ***n ≥ N***, for some integer ***N***;
3. .

**Homework 9.5c: page 523 # 18**

b. The Alternating Series Estimation Theorem

If an alternating series  satisfies the conditions

of the Alternating Series Test, then the truncation error for the **nth** partial sum is

less than ***un+1*** and has the same sign as the first unused term.

**Homework 9.5d: page 523 # 23, 24**

**Homework 9.5e: page 525 # 1 – 4.**