AP CALCULUS (AB) NAME ________________________

Outline – Chapter 4 Overview Date ___________________________

Objectives of Chapter 4

1) Using the derivative to determine extreme values of a function and the general shape of a function's graph (including where the function increases/decreases, inflection points, concavity, etc.);

2) "Recovering" a function from its derivatives and a single point;

3) Applying the concepts explored in determining extreme values of a function to modeling and optimization problems.

4) Using local linearity, we will see how the tangent line "captures" the shape of a curve near the point of tangency, and therefore provides a good approximation for the curve's function values as well as approximating change in function values;

5) Applying the concept of local linearity in determining roots of functions;

6) Determining rates of change we cannot measure, from rates of change we already know (a concept called related rates);

Section 4.1 – Extreme Values of Functions

1) Informal/Intuitive discussion of the graph of the function on page 189 Figure 4.4.

    i) Discuss Absolute (Global) Extrema;
    ii) Discuss Relative (Local) Extrema;
    iii) Ask students to describe a function's graph's behavior at an extrema;
    iv) Ask students to express their ideas in terms of the function's derivative.

2) Next draw the graphs of the same function defined on different intervals, including open intervals similar to page 188 Figure 4.2.

    i) Discuss how changing the intervals effects extrema;
    ii) Asks students under what conditions you can be guaranteed that both an absolute maximum and minimum will occur.

3) Now begin to formalize the ideas explored graphically with definitions and theorems:

    i) Absolute (global) maximum/minimum values
    ii) Extreme Value Theorem {f must be cont on closed interval}
    iii) Relative (local) maximum/minimum values
    iv) Local Extreme Values Theorem
    {local extreme value at interior pt where f' exist → $f'(c) = 0$}
    v) Critical point {pt $c \varepsilon D$ where $f'(c) = 0$ or D.N.E.}

Homework 4.1a: pages 193: # 1 - 10

4) Working Some Examples

i) Absolute Extrema – page 190 Example 3
ii) Extreme Value – page 190 Example 4
iii) Very Important Discussion about $y = x^3$ and $y = x^{1/3}$.
{Not every critical point or endpoint signals the presence of an extrema}
iv) Absolute Value Functions – page 192 Example 6.

Homework 4.1b: page 193 # 11, 13, 15, 17, 18, 19, 21, 23, 25, 29, 35, 42, 43, 44, 51, 52

Section 4.2 – The Mean Value Theorem (Derivatives)

The Mean Value Theorem makes an algebraic and graphical connection between the average rate of change for a function over an interval and the instantaneous rate of change for the function at a specific point in the interval. We will use it, and its corollaries, in many important applications.

Before stating and proving the Mean Value Theorem, we will state and prove a theorem which will be used in the proof of The MVT – Rolle's Theorem.

1) Rolle's Theorem: IF $y = f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), and if $f(a) = f(b) = 0$, THEN $\exists$ at least one $c \in (a, b)$ such that $f'(c) = 0$.

- Proof and Discuss graphical interpretation of Rolle's Theorem.

2) Mean Value Theorem: If $y = f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), THEN $\exists$ at least one $c \in (a, b)$ such that

$$\mathbf{f'(c) = \frac{f(b) - f(a)}{b - a}}.$$

- Proof
- Discuss graphical interpretation of The MVT Theorem (page 198, Example #4)

Homework 4.2a: page 202 # 1 – 11 odd

3) Definitions: Increasing/Decreasing Function

Let $\boldsymbol{f}$ be a function defined an interval, I, and let x_1 and x_2 be any two points in I.

i) we say that $\boldsymbol{f}$ <u>increases</u> on I if $x_1 < x_2 \rightarrow \boldsymbol{f}(x_1) < \boldsymbol{f}(x_2)$.
ii) we say that $\boldsymbol{f}$ <u>decreases</u> on I if $x_1 < x_2 \rightarrow \boldsymbol{f}(x_1) > \boldsymbol{f}(x_2)$.

We will now state and prove 3 very significant Corollaries to *The MVT for Derivatives*.

4) Corollary 1 (MVT) – Increasing/Decreasing Functions

Let f be continuous on [a, b] and differentiable on (a, b).

i) IF $f'(x) > 0$ for all $x \in (a, b)$, THEN f increases on [a, b].
ii) IF $f'(x) < 0$ for all $x \in (a, b)$, THEN f decreases on [a, b].

- Proof
- Application – Determining Where a Given Function Increases/Decreases (see page 199 Example 6)

Homework 4.2b: page 202 # 15, 18, 19, 22, 23, 26, 27, 39, 41, 42, 45

5) Corollary 2 (MVT) – The Derivative of a Constant Function

IF $f'(x) = 0$ for all $x \in I$,
THEN there exists a constant C for which $f(x) = C$, for all $x \in I$.

- Proof and Discussion (used to prove the next corollary)

6) Corollary 3 (MVT) – Derivatives of Functions Which Differ by a Constant

IF $f'(x) = g'(x)$ for all $x \in I$,
THEN there exists a constant C such that $f(x) = g(x) + C$ for all $x \in I$.

- Proof and Discussion
- Application – Recovering a Function from its Derivative and a Given Point (see page 200 Example 7)

7) Definition - Antiderivative

A function F(x) is an antiderivative of a function f(x) if $F'(x) = f(x)$ for all x in D_f. The process of determining and antiderivative is called antidifferentiation.

Homework 4.2c: page 202 # 31 – 34, 35, 37, 55, 58, 59, 60

Section 4.3 – Connecting *f'* *and* *f"* with the Graphs of *f*

We now formalize some of the concepts encountered in the previous two sections.

1) Local Extreme Values of a Function

a) Discuss the behavior of the graph of a function and extrema in relationship to the sign of the derivative – see page 205 Figure 4.18.
b) Theorem 4 - First Derivative Test for Local Extrema (page 205)
c) Application – page 206 Example 1a

Homework 4.3a: page 215 # 2, 4, 5, 6

2) Concavity

a) Definition: The graph of a differentiable function, **y = *f*(x)** is…

i) <u>concave up</u> on an open interval, ***I***, if ***f'*(x)** is increasing on ***I***.
ii) <u>concave down</u> on an open interval, ***I***, if ***f'*(x)** is decreasing on ***I***.

b) Concavity Test: The graph of a twice differentiable function, **y = *f*(x)** is

i) <u>concave up</u> on any interval where $f''(x) > 0$
ii) <u>concave down</u> on any interval, where $f''(x) < 0$.

c) Definition: A <u>point of inflection</u> is a point on the graph of a function where the tangent line exists and the concavity changes.

d) Applications
i) page 208 Example # 3 – Inflection Points and Concavity
ii) page 209 Example # 4 – Reading the Graph of $f'(x)$.
iii) Revisit page 206 Example 1a: Consider: $f(x) = x^3 - 12x - 5$ (using number line analysis)
iv) Another Important Discussion…$y = x^4$ ***and*** $y = x^{1/3}$
v) Motion Along a Line – page 209 Example # 5

Homework 4.3b: page 215 # 8, 11, 13, 16, 19, 21, 23, 27, 29, 57, 59

3) Second Derivative Test for Local Extrema

a) Theorem 5: If $f'(c) = 0$ and $f''(c) < 0$, then ***f*** has a local maximum at x = c.
If $f'(c) = 0$ and $f''(c) > 0$, then ***f*** has a local minimum at x = c.

b) Application – page 212 Example 8

Homework 4.3c: page 215 # 35, 38, 39, 41, 43, 44, 47, 51

4) Analyzing a Discontinuous Derivative – Page 214 Example # 9

Homework 4.3d: page 215 # 49

Review Exercises for Test # 1
Pages 256 – 260: # 1, 5, 8, 17, 19 – 23, 25, 26, 31 – 33, 35, 36, 37, 38, 70*

Section 4.4 – Modeling and Optimization

Often in practical situations, there is a need to optimize a situation. You might be trying to: 1) generate the most profit, 2) spend the least time on a task, or 3) determine the best dose of a medication. In either case, you are involved in a process that seeks to maximize or minimize some quantity. If those quantities can be expressed as functions, then you can apply the concepts of calculus to help determine where/when those maxima/minima occur, and what those values are. In order to optimize these quantities we will be applying the concepts of the early sections of chapter 4, primarily that of using the first derivative to identify where extrema occur.

1) The Steps of Solving Optimization Problems

   i) Determine whether the problem statement requires maximizing or minimizing;
   ii) Identify the quantity to be optimized;
   iii) Modeling: Express the quantity to be optimized as a function of a single variable (this will involve identifying the constraint equation and substitution);
   iv) Differentiate the function to be optimized, and use it to determine local extrema; consider the domain - does the domain involve endpoints?
   v) Do a sign "line" analysis to determine any absolute maxima/minima;
   vi) Answer the question!!!

   In addition to the above steps, it is assumed that you would read the problem statement carefully for a thorough understanding of the situation. It may be helpful to draw a diagram where appropriate.

2) Optimization Examples from Mathematics

   a) Numeric – Find two numbers whose sum is 20 and whose product is largest;
   b) Geometry – Given 100 feet of fencing, determine the dimensions of the rectangle that gives the greatest area;
   c) Inscribing a Rectangle – see Example 2 (page 219);
   d) Distance – see page 229 # 42;
   e) Determining the sign of a function – see page 229 # 44

Homework 4.4a: page 226 # 2, 3, 5, 6, 9, 41, 43, 46, 55, 56

3) Manufacturing a box – see Example 3 (page 221)

Homework 4.4b: page 226 # 7, 16, 17

   3) Economics

      a) Maximum Profit
         i) Theorem 6 – page 223;
         ii) Example 5 – page 224;

b) Minimizing Average Cost

i) Theorem 7 – page 224;

3) Example 6 – page 225;

Homework 4.4c: page 226 # 23, 25, 33, 34, 41, 50, 52, 53, 54, 61.

Section 4.5 – Linearization and Newton's Method

One of the most useful and frequent applications of the derivative is in the determination of approximations of "sought after" values. In this section we will investigate how the derivative (and its graphical representation – the tangent line) can be used to approximate: 1) roots; 2) function values; and 3) changes in function values. *Local linearity* is the concept that provides the foundation for these approximations by asserting that all differentiable functions can be well-represented by a line tangent to the curve.

3) Linearization – if f is differentiable at x = a, then f has a tangent line at x = a, whose

equation is given by:

$\mathbf{y - y_1 = m(x - x_1)}$ or $\mathbf{y = y_1 + m(x - x_1)}$

since $x_1 = a$, we let $y_1 = f(a)$; $m = f'(a)$; and $y = L(x)$;

L(x) = f(a) + f'(a)[x – a].	This equation and its graph (a line) define the linearization of the function, f, at x = a.

The approximation $\mathbf{f(x) \approx L(x)}$ is the linear approximation of f at a. The point x = a is the center of the approximation.

Example #1: Determine the linearization for $\mathbf{g(x) = \cos x}$ at $\mathbf{x = \pi/2}$, and use it to approximate **cos (1.75)**.

Example #2: Approximate $\sqrt{9.02}$ using a linearization.

Homework 4.5a: page 242 # 2, 3, 5, 7, 11, 13, 45, 59, 62, 71

2) Newton's Method (for Approximating Roots) – see Handout

3) Newton's Formula – see handout or board for demonstration and derivation of Newton's Formula;

b) Procedure for Newton's Method

i) reasonable first "guess" – "seed value";

ii) Repeated use of Newton's Formula until…;

Example: Solve $x^3 + 2x = -x - 1$

c) Newton's Formula and your graphing Calculator;
(repeat above example)

d) When does Newton's Method fail to provide the desired result?

Homework 4.5b: page 242 # 15, 57, 61, 63, 64, 65, 66 (BC only)

3) Differential **–** a "new" variable used to describe small changes in a pre-existing variable (see handout)

Let $y = f(x)$ be a differentiable function. The **differential dx** is an independent variable; The **differential dy** is a dependent variable: **dy = f'(x)* dx**.

Whereas Newton's Method was used to approximate roots/solutions of equations; and whereas a Linearization was used to approximate function values; Differentials are used to approximate the change in function values.

Consider dx to be a small change in x (usually $dx = \Delta x$);
then dy is the "small" change in y, given the specific x and dx above (usually $dy \approx \Delta y$).

If one considers that $f'(x) = dy/dx$ and interprets dy/dx as a ratio, then the above makes sense. (Although we know that dy/dx is not a ratio!!!)

3) Determining a Differential

Example: Let $y = \sin(3x)$, $x = \pi$, $dx = -.02$. Determine dy.

In summary, dy is a "function" of 3 parameters; dy depends on the <u>function</u>, the value of <u>x</u>, and the value of <u>dx</u>.

Homework 4.5c: page 242 # 19, 20, 21, 23, 25, 27, 29

b. Using Differentials to Estimate Change (see Handout)

For small changes in the value of x at a known point x = a, (dx), the change in the function value and the change in the *linearization* at x = a will be "close". In fact we could use ΔL to approximate Δf.

If **f** is differentiable at **x = a**, then the approximate change in the value of **f** when **x** changes from **a** to **a + dx** is given by: **df = f'(a) dx**.

Example # 8 (page 240)

Homework 4.5d: page 242 # 31, 33

c. Different Changes (Absolute, Relative, Percent) and Application

Homework 4.5e: page 242 # 35, 39, 43, 46, 49, 51, 60

Section 4.6 – Related Rates

This topic investigates a numerical relationship between the derivatives of three (or possibly more) quantities. The quantities are expressed as variables, and usually one of these variables is time. In order for the rates of these quantities to be related, the quantities themselves must be related. The concept of *Related Rates* is very closely tied to the Chain Rule for derivatives. We will begin our investigation with an example of how to problem solving using Related Rates.

Keep these steps (guidelines) in mind when solving:

i) Draw a diagram representing the problem description;
ii) Identify any given information into one of three quantities (constant throughout the problem, varying quantity, rate);
iii) Make a list of the variables involved;
iv) Identify the rate you are asked to determine;
v) Develop an equation relating the quantities involved;
vi) Determine the derivative of the above equation (be mindful of the Chain Rule);
vii) Substitute any appropriate given values into the above derivative equation;
viii) Solve for the desired quantity and interpret your results – be mindful of units!

3) See page 248 – Example #4 (Problem Twist: ________________________________)

Homework 4.6.1: page 251 # 16, 17, 20, 42, 38, 39

2) See page 247 – Example #2

Homework 4.6.2: page 251 # 11, 25, 31

3) See page 247 – Example #3 (Problem Twist: ________________________________)

Homework 4.6.3: page 251 # 9, 19, 22, 32

Review Exercises for Test # 2

Page 255 # 1 – 4;
Page 257 # 39, 40, 42, 58, 60, 61, 62, 66